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Strong Asymptotics for Orthogonal Polynomials with Regularly and Slowly Varying Recurrence Coefficients

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Contracted strong asymptotics away from the oscillatory region are given for orthogonal polynomials whose recurrence coefficients are regularly or slowly varying sequences which tend to infinity with n.

1. INTRODUCTION

Consider the equation

$$a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) = x p_n, \qquad n = 1, 2, ..., \tag{1.1}$$

where the a_n's and b_n's are regularly varying at infinity, i.e., there exists an increasing positive sequence $\{\lambda_n, n = 0, 1, ...,\}$ such that

$$\lim_{n \to \infty} \frac{a_n}{\lambda_n} = a > 0, \qquad \lim_{n \to \infty} \frac{b_n}{\lambda_n} = b \in \mathbb{R}, \tag{1.2}$$

with

$$\lim_{n \to \infty} n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha \geqslant 0. \tag{1.3}$$

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Furthermore we shall assume that $a_{n+1} > 0$, $b_n \in \mathbb{R}$, $n = 0, 1, 2, \ldots$, and

$$\lim_{n \to \infty} n \frac{(a_{n+1} - a_n)}{\lambda_n} = a\alpha, \qquad \lim_{n \to \infty} n \frac{(b_{n+1} - b_n)}{\lambda_n} = b\alpha. \tag{1.4}$$

Polynomials orthogonal with respect to Freud and Erdös type weight functions fall into this class and have been the subject of intense investigation, see Nevai [10, 11], Máté, Nevai, and Totik [9], Lubinsky and Saff [8], Lubinsky, Mhaskar, and Saff [7], Geronimo and Van Assche [4], Van Assche [15, 16], Rakhmanov [13], Lubinsky [5, 6], and Smith [14]. Here we use the techniques partially developed in Van Assche and Geronimo [17] to give strong asymptotics of polynomials whose coefficients satisfy the conditions given by Eqs. (1.2)–(1.4). We proceed as follows: In Section 2 we review the results needed from Geronimo and Smith [3]. A useful result developed in [3] is the construction of continuations of the coefficients from the integers to all positive x having smooth derivatives. In Section 3, using the Euler–Maclaurin formula and the theory of regularly varying functions, we develop the asymptotic formulas mentioned above. Finally, in Section 4 a number of examples are explicitly worked out.

2. Preliminaries

Given sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ satisfying (1.2)–(1.4) we find from the theory of regularly varying sequences (Bojanic and Seneta [2]) that a_n, b_n, and λ_n can be represented as

$$a_n = an^{\alpha} l_1(n), \qquad b_n = bn^{\alpha} l_2(n), \qquad \text{and} \qquad \lambda_n = n^{\alpha} l(n), \tag{2.1}$$

where

$$\lim_{n \to 0} \frac{l_1(n)}{l(n)} = \lim_{n \to \infty} \frac{l_2(n)}{l(n)} = 1 \tag{2.2}$$

and

$$\lim_{n \to \infty} n \left(\frac{l(n+1)}{l(n)} - 1 \right) = \lim_{n \to \infty} n \left(\frac{l_1(n+1)}{l_1(n)} - 1 \right) = \lim_{n \to \infty} n \left(\frac{l_2(n+1)}{l_2(n)} - 1 \right) = 0. \tag{2.3}$$

The functions $l(n)$, $l_1(n)$, and $l_2(n)$ are called slowly varying functions. It further follows from the theory of regularly varying sequences [2, Theorem 4] that

$$\lim_{n \to \infty} \frac{a_{[nt]}}{a_n} = t^{\alpha} \qquad \text{and} \qquad \lim_{n \to \infty} \frac{b_{[nt]}}{b_n} = t^{\alpha}, \; b \neq 0 \tag{2.4}$$

for every $t>0$ and this coupled with (1.4) implies that

$$\lim_{n\to\infty} n\left(\frac{a_{[nt]+1}-a_{[nt]}}{\lambda_n}\right)=a\alpha t^{\alpha-1}, \tag{2.5}$$

and

$$\lim_{n\to\infty} n\left(\frac{b_{[nt]+1}-b_{[nt]}}{\lambda_n}\right)=b\alpha t^{\alpha-1}, \tag{2.6}$$

for all $t>0$.

In what follows we often need the interval

$$[D, E]=\text{convex hull}(\{0\}, [b-2a, b+2a]), \tag{2.7}$$

and always consider $y\notin[D, E]$.

LEMMA 2.1 [3, Lemma 3.2]. *Suppose* (1.2) *and* (1.3) *hold with* $a_i\to\infty$ *and* $|b_i|\to\infty$ *or* $b=0$. *If* $\alpha=0$ *we further suppose that*

$$\limsup\left\{\max_{1\leqslant k\leqslant n}\frac{a_k}{a_n}\right\}=1=\limsup\left\{\max_{0\leqslant k\leqslant n}\frac{b_k}{b_n}\right\}, \qquad b\neq 0. \tag{2.8}$$

Then for $y\notin[D, E]$ *there exists a* $d>1$ *such that for n sufficiently large*

$$|u_0(\lambda_n y, i)|>d, \qquad i=1, 2, ..., n, \tag{2.9a}$$

$$|v_0(\lambda_n y, i)|<1/d, \qquad i=1, 2, ..., n, \tag{2.9b}$$

where

$$u_0(x, i)=\frac{x-b_i}{2a_i}+\sqrt{\left(\frac{x-b_i}{2a_i}\right)^2-1}=\frac{1}{v_0(x, i)}. \tag{2.10}$$

Consider now the continuous extensions of l_1 and l_2 given by

$$\begin{aligned} L_i(x)={}&(l_i(n+2)-2l_i(n+1)+l_i(n))(x-n)^3-(x-n)^2)\\ &+(l_i(n+1)-l_i(n))(x-n)+l_i(n), \qquad n\leqslant x\leqslant n+1, i=1, 2, \end{aligned} \tag{2.11}$$

where for $L_1(x)$ we take $n\geqslant 1$ while for $L_2(x)$ we let $n\geqslant 0$.

LEMMA 2.2 [3, Lemma 3.7]. *Let* L_1 *and* L_2 *be as above, then* L_1 *and* L_2 *are* C^1 *slowly varying functions*, $\lim_{x\to\infty}(L_1(x)/l_1([x]))=1=\lim_{x\to\infty}(L_2(x)/l_2([x]))$. *If*

$$R_1(x)=x^\alpha L_1(x), \qquad \alpha\geqslant 0, \tag{2.12}$$

and

$$R_2(x) = x^\alpha L_2(x), \qquad \alpha \geqslant 0, \tag{2.13}$$

where α *is given by* (1.3) *then*

$$a_n = aR_1(n), \tag{2.14}$$

and

$$b_n = bR_2(n). \tag{2.15}$$

If $\alpha > 0$ *then*

$$\frac{1}{\lambda_n^2} \int_M^n |R_i'(y)|^2 \, dy = o(1) = \frac{1}{\lambda_n} \int_M^n |R_i''(y)| \, dy, \qquad i = 1, 2. \tag{2.16}$$

If $\alpha = 0$ *and*

$$\frac{1}{\lambda_n} \sum_{i=1}^n |a_{i+2} - 2a_{i+1} + a_i| = o(1) = \frac{1}{\lambda_n} \sum_{i=1}^n |b_{i+2} - 2b_{i+1} + b_i|, \tag{2.17}$$

then (2.16) *is still valid.*

With the above results we can prove the following.

THEOREM 2.3 [3, Theorem 3.8]. *Suppose* $y \notin [D, E]$ *and* (1.2) *through* (1.4) *hold with* $a_i \to \infty$ *and* $|b_i| \to \infty$ *or* $b = 0$. *If* $\alpha > 0$ *or for* $\alpha = 0$ *if* (2.8) *and* (2.17) *hold then*

$$\lim_{n \to \infty} \frac{p_n(\lambda_n y)}{\prod_{i=1}^n u_0(\lambda_n y, i)} = \left\{ \frac{(x-b)^2 - 4a^2}{x^2} \right\}^{-1/4} \exp\left\{ \frac{b}{2} \int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2 s^2}} \right\}, \tag{2.18}$$

uniformly on compact subsets of $\mathbb{C} \setminus [D, E]$.

3. ASYMPTOTIC FORMULAS

Theorem 2.3 allows us to compute strong asymptotics for $p_n(\lambda_n y)/\prod_{i=1}^n u_0(\lambda_n y, i)$. We now develop some formulas that help in the evaluation of $\prod_{i=1}^n u_0(\lambda_n y, i)$.

LEMMA 3.1. *Suppose* $y \notin [D, E]$ *and* (1.2) *through* (1.4) *hold. If* $\alpha > 0$ *and*

$$\lim_{n \to \infty} n \frac{(R_1(n) - R_2(n))}{\lambda_n} = \alpha A, \tag{3.1}$$

where

$$a_n = aR_1(n) \qquad \text{and} \qquad b_n = bR_2(n), \tag{3.2}$$

then

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{u_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)} = \exp\left(bA \int_0^1 \frac{dz}{\sqrt{(y-bz)^2 - 4a^2z^2}}\right), \tag{3.3}$$

uniformly on compact subsets of $\mathbb{C}\setminus[D, E]$. *Here*

$$\hat{u}_0(\lambda_n y, i) = \frac{\lambda_n y - bR_1(i)}{2aR_1(i)} + \sqrt{\left(\frac{\lambda_n y - bR_1(i)}{2aR_1(i)}\right)^2 - 1}. \tag{3.4}$$

Proof. We write

$$\sum_{i=1}^{n-1} \ln\left(\frac{u_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)}\right) = n \int_{1/n}^1 \ln\left(1 + \frac{u_0(\lambda_n y, [nt]) - \hat{u}_0(\lambda_n y, [nt])}{\hat{u}_0(\lambda_n y, [nt])}\right) dt,$$

where $[x]$ is the integer part of x. The dominated convergence theorem, (3.1), and the fact that

$$\lim_{n\to\infty} nd_n = d \Rightarrow \lim_{n\to\infty} n \ln(1 + d_n) = d,$$

imply that

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{u_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)} = \exp bA \int_0^1 \alpha \left(1 + \frac{y - bt^\alpha}{\sqrt{(y-bt^\alpha)^2 - 4a^2t^{2\alpha}}}\right)$$
$$\times \frac{t^{\alpha-1}\, dt}{(y - bt^\alpha) + \sqrt{(y-bt^\alpha)^2 - 4a^2t^{2\alpha}}}.$$

Setting $z = t^\alpha$ gives the result. ∎

In order to treat the case $\alpha = 0$ we need to make some stronger assumptions on the recurrence coefficients.

LEMMA 3.2. *Suppose* $\alpha = 0, y \notin [D, E]$, (1.2) *through* (1.4) *hold with* $a_i \to \infty$ *and* $|b_i| \to \infty$. *Suppose* (2.8) *and* (2.17) *hold, that* a_n *is strictly increasing for n sufficiently large, and that*

$$\lim_{n\to\infty} \frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} = 1. \tag{3.5}$$

If

$$\lim_{n\to\infty} \frac{R_1(n) - R_2(n)}{R_1(n+1) - R_1(n)} = B, \tag{3.6}$$

where $R_1(n)$ and $R_2(n)$ are given by (3.2), *then*

$$\lim_{n\to\infty}\prod_{i=1}^{n}\frac{u_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)}=\exp\left(bB\int_0^1\frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}}\right), \tag{3.7}$$

where the convergence is uniform on compact subsets of $\mathbb{C}\setminus[D, E]$.

Proof. Let $R_1(x)$ and $R_2(x)$ be the extensions given by Lemma 2.2 then

$$\begin{aligned} aR_1'(x) &= (a_{n+2}-2a_{n+1}+a_n)(3(x-n)^2-2(x-n))+(a_{n+1}-a_n)\\ &=(a_{n+1}-a_n)\left\{\left(\frac{a_{n+2}-a_n}{a_{n+1}-a_n}-1\right)(3(x-n)^2-2(x-n))+1\right\},\\ &\qquad n\leqslant x\leqslant n+1. \end{aligned}$$

Since $0\leqslant(x-n)\leqslant 1$ for $n\leqslant x\leqslant n+1$ and a_n is monotonically increasing for n sufficiently large we see that $R_1'(x)>0$ for x sufficiently large. Thus R_1^{-1} exists. Now write

$$\prod_{i=1}^{n}\frac{u_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)}=\exp\sum_{i=1}^{n}\ln(1+\Gamma(\lambda_n y, i)), \tag{3.8}$$

where

$$\Gamma(\lambda_n y, i)=\frac{u_0(\lambda_n y, i)-\hat{u}_0(\lambda_n y, i)}{\hat{u}_0(\lambda_n y, i)}. \tag{3.9}$$

From Lemma 2.1 we find for any compact set $K\subset\mathbb{C}\setminus[D, E]$ an N_0 such that $|\Gamma(\lambda_n y, i)|<1$ for all $n\geqslant N_0$ and $i=1, 2, ..., n$. Therefore

$$\sum_{i=1}^{n}\ln(1+\Gamma(\lambda_n y, i))=\sum_{i=1}^{n}\left(\Gamma(\lambda_n y, i)-\Gamma^2(\lambda_n y, i)\int_0^1\frac{\beta\, d\beta}{1+\beta\Gamma(\lambda_n y, i)}\right), \tag{3.10}$$

where $n\geqslant N_0$. From (3.9) we see that

$$\begin{aligned} |\Gamma(\lambda_n y, i)| &= \frac{b}{\lambda_n}O(|R_1(i)-R_2(i)|)\\ &=\frac{b}{\lambda_n}O\left(\left|\frac{R_1(i)-R_2(i)}{R_1(i+1)-R_1(i)}\right||R_1(i+1)-R_1(i)|\right), \end{aligned}$$

and this coupled with Potter's bound [1, p. 25], (3.6), and (1.4) imply that the second term in (3.10) is $o(1)$. Set

$$Q_1(\lambda_n y, i)=\sqrt{\left(y-\frac{bR_1(i)}{\lambda_n}\right)^2-4a^2\frac{R_1(i)^2}{\lambda_n^2}}, \tag{3.11}$$

and

$$\Gamma_1(\lambda_n y, i) = \frac{b(R_1(i) - R_2(i))}{2aR_1(i)\, \hat{u}_0(\lambda_n y, i)} \left[1 + \frac{\lambda_n y - bR_1(i)}{\lambda_n Q_1(\lambda_n y, i)} \right]. \tag{3.12}$$

Now (2.9) implies that

$$\begin{aligned} |\Gamma(\lambda_n y, i) - \Gamma_1(\lambda_n y, i)| &= O((R_2(i) - R_1(i))^2/\lambda_n) \\ &= O\left[\left(\frac{R_2(i) - R_1(i)}{R_1(i+1) - R_1(i)} \right)^2 \frac{(R_1(i+1) - R_1(i))^2}{\lambda_n} \right]. \end{aligned}$$

Therefore Potter's bound, (1.4), (3.6), (3.10) and the above equation imply that

$$\sum_{i=1}^{n} \ln(1 + \Gamma(\lambda_n y, i)) = \sum_{i=1}^{n} \Gamma_1(\lambda_n y, i) + o(1).$$

An application of the Euler–Maclaurin formula (Olver [12, p. 285]) yields

$$\begin{aligned} \sum_{i=1}^{n} \Gamma_1(\lambda_n y, i) = \sum_{i=1}^{M-1} \Gamma_1(\lambda_n y, i) &+ \int_M^n \Gamma_1(\lambda_n y, x)\, dx \\ &+ \frac{1}{2} \Gamma_1(\lambda_n y, n) + \frac{1}{2} \Gamma_1(\lambda_n y, M) \\ &+ \int_M^n (B_2 - B_2(x - [x])) \frac{d^2}{dx^2} \Gamma_1(\lambda_n y, x)\, dx, \end{aligned} \tag{3.13}$$

where $\Gamma_1(\lambda_n y, x)$ is given in (3.12) with i replaced by x and M is chosen sufficiently large. From (2.9) and (2.11) we find that there exists a $C > 0$ such that $|Q_1(\lambda_n y, x)| > C$ for $y \notin [D, E]$, $0 \leqslant x \leqslant n$, and n sufficiently large. Therefore (1.4) and (3.6) imply that the third and fourth terms in (3.13) are $o(1)$. For fixed M the same is true of the first term in (3.13). Consequently

$$\begin{aligned} \sum_{i=1}^{n} \Gamma_1(\lambda_n y, i) = \int_M^n \Gamma_1(\lambda_n y, x)\, dx & \\ + \int_M^n (B_2 - B_2(x - [x])) \frac{d^2}{dx^2} \Gamma_1(\lambda_n y, x)\, dx + o(1). & \end{aligned} \tag{3.14}$$

Now differentiate $\Gamma_1(\lambda_n y, x)$ twice with respect to x. Since $|Q_1(\lambda_n y, x)| > C$ for $y \notin [D, E]$, $0 \leqslant x \leqslant n$, n sufficiently large,

$$\left| \frac{d^2}{dx^2} \Gamma_1(\lambda_n y, x) \right| = O\left(\frac{|R_1''(x)|}{\lambda_n} + \frac{|R_2''(x)|}{\lambda_n} + \frac{|R_1'(x)|^2}{\lambda_n^2} + \frac{|R_2'(x)|^2}{\lambda_n^2} \right).$$

Therefore by Lemma 2.2 we see that the second integral on the right hand side of (3.14) is $o(1)$. Choose M sufficiently large so that $R_1(x)$ is strictly increasing for $x \geqslant M$, then

$$\begin{aligned}\int_M^n \Gamma_1(\lambda_n y, x)\, dx &= \int_M^n \frac{\Gamma_1(\lambda_n y, x)}{R_1(x+1)-R_1(x)} (R_1(x+1)-R_1(x))\, dx \\ &= \int_M^n \frac{\Gamma_1(\lambda_n y, x)}{R_1(x+1)-R_1(x)} R_1'(x)\, dx \\ &\quad + \int_M^n \frac{\Gamma_1(\lambda_n y, x)}{R_1(x+1)-R_1(x)} \int_x^{x+1} (x+1-y)\, R_1''(y)\, dy.\end{aligned} \tag{3.15}$$

From (3.6) and the the fact Q_1 is bounded away from zero we observe that

$$\left|\int_M^n \frac{\Gamma_1(\lambda_n y, x)}{R_1(x+1)-R_1(x)} \int_x^{x+1} R_1''(y)\, dy\right| \leqslant \frac{C}{\lambda_n} \int_M^n \int_x^{x+1} |R_1''(y)|\, dy = o(1),$$

where Lemma 2.2 has been used to obtain the last inequality. Setting $z = R_1(x)$ in (3.15) and using the fact that $R_1(x)$ has a unique inverse for $x \geqslant M$ yields

$$\int_M^n \Gamma_1(\lambda_n y, x)\, dx = \frac{b}{\lambda_n} \int_{R_1(M)}^{R_1(n)} \frac{f(z)\, dz}{\sqrt{(y-bz/\lambda_n)^2 - 4a^2 z^2/\lambda_n^2}} + o(1), \tag{3.16}$$

with

$$f(z) = \frac{R_1(R_1^{-1}(z)) - R_2(R_1^{-1}(z))}{R_1(R_1^{-1}(z)+1) - R_1(R_1^{-1}(z))}. \tag{3.17}$$

Equation (3.5) implies that

$$\lim_{n\to\infty} \frac{R_1(n+1)-R_2(n+1)}{R_1(n+1)-R_1(n)} = \lim_{n\to\infty} \frac{R_1(n+2)-R_2(n+2)}{R_1(n+1)-R_1(n)} = B,$$

and since

$$\begin{aligned}&\frac{R_1(x+1)-R_1(x)}{R_1(n+1)-R_1(n)} \\ &\quad = \left(\frac{R_1(n+3)-R_1(n+2)}{R_1(n+1)-R_1(n)} - \frac{2(R_1(n+2)-R_1(n+1))}{R_1(n+1)-R_1(n)} + 1\right) \\ &\qquad \times ((x-n)^3 - (x-n)^2) \\ &\qquad + \left(\frac{R_1(n+2)-R_1(n+1)}{R_1(n+1)-R_1(n)} - 1\right)(x-n) + 1, \qquad n \leqslant x \leqslant n+1,\end{aligned}$$

by (2.11) we see that

$$\lim_{x\to\infty} \frac{R_1(x)-R_2(x)}{R_1(x+1)-R_1(x)} = B. \tag{3.18}$$

Set $z = R_1(n)\,u$ in (3.16) to find

$$\frac{b}{\lambda_n}\int_{R_1(M)}^{R_1(n)} \frac{f(z)\,dz}{\sqrt{(y-bz/\lambda_n)^2-4a^2z^2/\lambda_n^2}}$$
$$=\frac{bR_1(n)}{\lambda_n}\int_{R_1(M)/R_1(n)}^{1} \frac{f(R_1(n)\,u)\,du}{\sqrt{(y-(bR_1(n)/\lambda_n)\,u)^2-4a^2R_1(n)^2\,u^2/\lambda_n^2}}.$$

Since $\lim_{n\to\infty} R_1^{-1}(R_1(n)\,u)=\infty$ for every $u>0$, (3.18), (1.2), and the dominated convergence theorem give the result. ∎

We shall now develop a formula for $\prod_{i=1}^n \hat{u}_0(\lambda_n y, i)$.

LEMMA 3.3. *Let $R_1(x), x>0$, be any C^1 extension of $R_1(n)$ where $a_n = aR_1(n)$ $(n=1, 2, ...)$ such that* (2.16) *is satisfied. Then for $y \notin [D, E]$*

$$\lim_{n\to\infty} \frac{\prod_{i=1}^n (a_i/a\lambda_n)\,\hat{u}_0(\lambda_n y, i)}{u(R_1(n)/\lambda_n)^{n+1/2}\,K(n)} = \sqrt{\frac{a}{y}},$$

where the convergence is uniform on compact subsets of $\mathbb{C}\setminus[D, E]$. Here

$$u\left(\frac{R_1(n)}{\lambda_n}\right)=\frac{y-bR_1(n)/\lambda_n}{2a}+\sqrt{\left(\frac{y-bR_1(n)/\lambda_n}{2a}\right)^2-\left(\frac{R_1(n)}{\lambda_n}\right)^2}, \tag{3.19}$$

and

$$K(n)=\exp\int_0^n x\left(\frac{y}{\sqrt{(y-bR_1(x)/\lambda_n)^2-4a^2((R_1(x))/\lambda_n)^2}}-1\right)\frac{R_1'(x)}{R_1(x)}\,dx. \tag{3.20}$$

If $R_1(x)$ has an inverse for $x \geqslant 0$, then the above integral can be recast into

$$K(n)=\exp\int_{R_1(0)/R_1(n)}^{1} R_1^{-1}(R_1(n)\,z)$$
$$\times\left(\frac{y}{\sqrt{(y-bR_1(n)\,z/\lambda_n)^2-4a^2(R_1(n)^2\,z/\lambda_n^2)}}-1\right)\frac{dz}{z}. \tag{3.21}$$

Proof. We begin by writing

$$\prod_{i=1}^n \frac{a_i}{a\lambda_n}\,\hat{u}_0(\lambda_n y, i)=\exp\sum_{i=1}^n \ln\frac{a_i}{a\lambda_n}\,\hat{u}_0(\lambda_n y, i).$$

Using the Euler–Maclaurin formula we find

$$\sum_{i=1}^{n} \ln \frac{a_i}{a\lambda_n} \hat{u}_0(\lambda_n y, i)$$

$$= \int_1^n \ln f(x)\, dx + \frac{1}{2} \ln \frac{a_n}{a\lambda_n} \hat{u}_0(\lambda_n y, n) + \frac{1}{2} \ln \frac{a_1}{a\lambda_n} \hat{u}_0(\lambda_n y, 1)$$

$$+ \int_1^n \frac{B_2 - B_2(x - [x])}{2} \frac{d^2}{dx^2} \ln f(x)\, dx, \tag{3.22}$$

where

$$f(x) = y - \frac{bR_1(x)}{\lambda_n} + \sqrt{\left(y - \frac{bR_1(x)}{\lambda_n}\right)^2 - \frac{4a^2 R_1^2(x)}{\lambda_n^2}}. \tag{3.23}$$

Now

$$\frac{f'(x)}{f(x)} = \left(-\frac{b}{Q_1(x)} - \frac{4a^2 R_1(x)}{\lambda_n Q_1(x) f(x)} \right) \frac{R_1'(x)}{\lambda_n}, \tag{3.24}$$

and since $|Q_1(x)| > 0$ and $|f(x)| > 0$ for n sufficiently large we find by differentiating (3.24) that

$$\left| \frac{d^2}{dx^2} \ln f(x) \right| = O\left(\frac{|R_1''(x)|}{\lambda_n} + \frac{R_1'(x)^2}{\lambda_n^2} \right).$$

Equation (2.16) and the above equation imply that

$$\left| \int_1^n \frac{B_2 - B_2(x - [x])}{2} \frac{d^2}{dx^2} \ln f(x)\, dx \right| = o(1). \tag{3.25}$$

If we integrate the first integral on the RHS of (3.22) by parts we find

$$\int_1^n \ln f(x)\, dx = n \ln\left(2au\left(\frac{R_1(n)}{\lambda_n}\right)\right) - \ln\left(2au\left(\frac{R_1(1)}{\lambda_n}\right)\right)$$

$$+ \int_1^n x \left(\frac{y}{Q_1(x)} - 1 \right) \frac{R_1'(x)}{R_1(x)}\, dx. \tag{3.26}$$

To arrive at the last integral we have used the fact that

$$\frac{1}{f(x)} = \frac{y - bR_1(x)/\lambda_n - \sqrt{(y - bR_1(x)/\lambda_n)^2 - 4a^2 (R_1(x)/\lambda_n^2)^2}}{4a^2 (R_1(x)/\lambda_n^2)^2}.$$

For large n we find that

$$\left(\frac{y}{Q_1(x)}-1\right)=\frac{b}{y}\frac{R_1(x)}{\lambda_n}+O\left(\left(\frac{R_1(x)}{\lambda_n}\right)^2\right),$$

which implies that

$$\int_1^n x\left(\frac{y}{Q_1(x)}-1\right)\frac{R_1'(x)}{R_1(x)}\,dx=\int_0^n x\left(\frac{y}{Q_1(x)}-1\right)\frac{R_1'(x)}{R_1(x)}\,dx+o(1). \tag{3.27}$$

Combining (3.27), (3.26), (3.25), and the fact that

$$\lim_{n\to\infty} u\left(\frac{R_1(1)}{\lambda_n}\right)=\frac{y}{a}=\lim_{n\to\infty}\frac{R_1(1)}{\lambda_n}\hat{u}_0(\lambda_n y, 1)$$

gives the result with $K(n)$ given by (3.20). If $R_1^{-1}(x)$ exists for $x\geqslant 0$ then the change of variables $z=R_1(x)/R_1(n)$ gives (3.21). ∎

Remark 3.4. If $R_1(x)$ is a C^1 extension of $R_1(n)$ so that (2.16) is satisfied and there exists an integer M such that $R_1^{-1}(x)$ exists for all $x\geqslant M$ then (3.21) is still true if we replace $R_1(0)$ by $R_1(M)$.

Although in the above lemma we have used the exact inverse of R_1, i.e., R_1^{-1} such that for all $x\geqslant 0$, $R_1^{-1}(R_1(x))=x$, it may be more convenient to use a different asymptotic inverse. We say that R_1^* is an asymptotic inverse of R_1 if $\lim_{x\to\infty}(R_1^*(R_1(x))/x)=1$. This implies that R_1^* and R_1^{-1} are asymptotically equivalent.

LEMMA 3.5. *Let $R_1^*(x)$, defined for $x\geqslant R_1(0)$, be an asymptotic inverse of $R_1(x)$. If*

$$\lim_{x\to\infty}|R_1^*(R_1(x))-x|=0 \tag{3.28}$$

then

$$\lim_{n\to\infty}\frac{K(n)}{K^*(n)}=1,$$

where $K(n)$ is given by (3.21) *and $K^*(n)$ is given by the same formula with R_1^{-1} replaced by R_1^*.*

Proof. From (3.20) we find that

$$\frac{K^*(n)}{K(n)}-1=\exp\left[\int_{R_1(0)/R_1(n)}^1 (R_1^*(R_1(n)\,z)-R_1^{-1}(R_1(n)\,z))\,g_n(z)\right]dz-1,$$

where

$$g_n(z)=\left(\frac{y}{\sqrt{(y-bR_1(n)\,z/\lambda_n)^2-4a^2(R_1^2(n)\,z^2/\lambda_n^2)}}-1\right)\frac{1}{z}.$$

Since (3.28) implies that $\lim_{x\to\infty}|R_1^*(x)-R_1^{-1}(x)|=0$ and since $g_n(z)$ is integrable for all n the result follows from the dominated convergence theorem. ▌

We will now obtain a formula for $\prod_{i=1}^n a_i/\lambda_n$. Before doing this, however, we need one more technical lemma.

LEMMA 3.6. *Suppose a_n satisfies* (1.2) *and* (1.4) *and $a_n\to\infty$. Let $R_1(x)$ be the extension given by Lemma* 2.2. *If $\alpha>0$ or if $\alpha=0$ and*

$$\sum_{i=1}^{\infty}\left|\frac{a_{i+2}-2a_{i+1}+a_i}{a_i}\right|<\infty, \tag{3.29}$$

then

$$\int_n^\infty\left|\frac{R'(x)}{R(x)}\right|^2 dx=o(1)=\int_0^\infty\left|\frac{R''(x)}{R(x)}\right| dx. \tag{3.30}$$

Proof. From (2.12) we see that

$$R'(x)=\frac{\alpha R(x)}{x}+x^\alpha L'(x),$$

and

$$R''(x)=\alpha(\alpha-1)\frac{R(x)}{x^2}+\alpha x^{\alpha-1}L'(x)+x^\alpha L''(x). \tag{3.31}$$

Consequently we find that

$$\int_n^\infty\left|\frac{R'(x)}{R(x)}\right|^2 dx\leqslant 2\left(\alpha\int_n^\infty\frac{dx}{x^2}+\sum_{i=n}^{\infty}\int_i^{i+1}\left|\frac{L'(x)}{L(x)}\right|^2 dx\right). \tag{3.32}$$

Now (2.11) tells us that

$$\frac{L'(x)}{L(x)}=\frac{((c_{i+1}-c_i)/l_1(i))(3(x-i)^2-2(x-i))+c_i/l_1(i)}{((c_{i+1}-c_i)/l_1(i))((x-i)^3-(x-i)^2)+(c_i/l_1(i))(x-i)+1},\quad i\leqslant x\leqslant i+1, \tag{3.33}$$

where

$$c_i=l_1(i+1)-l_1(i). \tag{3.34}$$

Since $c_i/l_1(i) \to 0$ as $i \to \infty$ by (2.3) we see that $|L(x)| > r > 0$ for x large enough. Therefore

$$\int_i^{i+1} \left|\frac{L'(x)}{L(x)}\right|^2 dx = O\left(\left(\frac{l_1(i+2)}{l_1(i+1)} - 1\right)^2 + \left(\frac{l_1(i+1)}{l_1(i)} - 1\right)^2\right),$$

which implies via (2.3) that the second term in (3.32) is $o(1)$. To show the second part of (3.30) we note that from (3.31) we find

$$\left|\frac{R''(x)}{R(x)}\right| \leqslant \frac{\alpha(\alpha-1)}{x^2} + \alpha\left|\frac{L'(x)}{xL(x)}\right| + \left|\frac{L''(x)}{L(x)}\right|. \tag{3.35}$$

From (3.33) and (3.34) we see that the second term in (3.35) is

$$\left|\frac{L'(x)}{xL(x)}\right| = O\left(\frac{1}{i}\left[\left|\frac{l_1(i+2)}{l_1(i+1)} - 1\right| + \left|\frac{l_1(i+1)}{l_1(i)} - 1\right|\right]\right), \qquad i \leqslant x \leqslant i+1.$$

Therefore

$$\int_n^\infty \left|\frac{L'(x)}{xL(x)}\right| dx = \sum_{i=n}^\infty \int_i^{i+1} \left|\frac{L'(x)}{xL(x)}\right| dx = o(1) \tag{3.36}$$

from (2.3). Since

$$\left|\frac{L''(x)}{L(x)}\right| \leqslant C\left|\frac{c_{i+1}-c_i}{l_1(i)}\right|, \qquad i \leqslant x \leqslant i+1,$$

for i large enough (3.34) implies

$$\left|\frac{c_{i+1}-c_i}{l_1(i)}\right| = O\left(\left|\frac{a_{i+2}-2a_{i+1}+a_i}{a_i}\right| + \left|\frac{a_{i+2}-a_{i+1}}{ia_i}\right|\right).$$

Thus from (2.1), (2.3), and if $\alpha = 0$ (3.29),

$$\int_n^\infty \left|\frac{L''(x)}{L(x)}\right| dx = \sum_{i=n}^\infty \int_i^{i+1} \left|\frac{L''(x)}{L(x)}\right| dx = o(1).$$

Consequently the lemma is proved. ∎

With this we can prove the following lemma.

LEMMA 3.7. *Let $a_n = aR_1(n)$ and suppose $R_1(x)$ is any C^1 extension of $R_1(n)$ such that $R_1(x) \neq 0$ for all x greater than or equal to one and such* *that* (3.30) *holds.* (*The second condition holds for the extension given by Lemma* 2.2 *provided* $\alpha > 0$ *or for* $\alpha = 0$ *provided* (3.29) *holds.*) *Then*

$$\lim_{n\to\infty} \frac{A(n)\prod_{i=1}^{n}(a_i/\lambda_n)}{(a_n/\lambda_n)^{n+1/2}(a_1/\lambda_n)^{-1/2}} = \exp\left(\frac{1}{2}\int_1^{\infty}(B_2 - B_2(x-[x]))\frac{d^2}{dx^2}\ln R_1(x)\,dx\right). \tag{3.37}$$

Here

$$A(n) = \exp\int_1^n x\frac{R_1'(x)}{R_1(x)}\,dx. \tag{3.38}$$

Proof. By means of the Euler–Maclaurin formula we have

$$\begin{aligned}\prod_{i=1}^{n}\frac{a_i}{\lambda_n} &= \left(\frac{a}{\lambda_n}\right)^n \exp\sum_{i=1}^{n}\ln R_1(i)\\ &= \left(\frac{a}{\lambda_n}\right)^n \exp\left(\int_1^n \ln R_1(x)\,dx + \frac{1}{2}\ln R_1(n) + \frac{1}{2}\ln R_1(1)\right.\\ &\quad \left. + \frac{1}{2}\int_1^n (B_2 - B_2(x-[x]))\frac{d^2}{dx^2}\ln R_1(x)\,dx\right).\end{aligned} \tag{3.39}$$

Integrate the first integral by parts to find

$$\int_1^n \ln R_1(x)\,dx = n\ln R_1(n) - \ln R_1(1) - \int_1^n x\frac{R_1'(x)}{R_1(x)}\,dx.$$

From (3.30) it follows that the integral

$$\int_1^{\infty}(B_2 - B_2(x-[x]))\frac{d^2}{dx^2}\ln R_1(x)\,dx$$

converges, which gives the desired result. ∎

Remark. If $R_1(n)$ has an analytic extension satisfying the hypothesis of the above lemma then one may be able to obtain other integral representations for the term on the right hand side of (3.37) (see Olver [12, p. 291]).

Combining these results leads to the two main theorems of this section.

THEOREM 3.8. *Suppose* (1.2) *and* (1.4) *are satisfied with* $\alpha > 0$. *If* $a_n = aR_1(n)$ *and* $b_n = bR_2(n)$ *are such that* (3.1) *holds, if* $R_1(x)$, $x \geqslant 0$, *is a*

C^1 extension of $R_1(n)$ such that $R_1(x)>0$ for $x \geqslant 1$, and if (2.16) *and* (3.30) *hold, then*

$$\lim_{n\to\infty} \frac{p_n(\lambda_n y)}{((a\lambda_n/a_n)\, u(R_1(n)/\lambda_n))^n\, K(n)\, A(n)(a_1/a\lambda_n)^{1/2}} = \frac{R\sqrt{au(R_1(n)/\lambda_n)}}{((y-b)^2-4a^2)^{1/4}} \exp\left\{b\left(A+\frac{1}{2}\right)\int_0^1 \frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}}\right\},$$

where the convergence is uniform on compact subsets of $\mathbb{C}[D, E]$. Here

$$R=\exp\left(-\int_1^\infty \frac{(B_2-B_2(x-[x]))}{2}\frac{dx^2}{dx^2}\ln R_1(x)\,dx\right),$$

$A(n)$ is given by (3.38), *$K(n)$ by* (3.20), *and $u(R_1(n)/\lambda_n)$ by* (3.19). *If $R_1(x)$ has an inverse for $x \geqslant 0$ then* (3.21) *may be used in place of* (3.20). *If $R_1^*(x)$ is defined for $x \geqslant 0$ and is any other asymptotic inverse such that* (3.28) *holds then $K(n)$ may be replaced by $K^*(n)$ where*

$$K^*(n)=\exp\int_{R_1(0)/R_1(n)}^1 R_1^*(R_1(n)\,z) \times\left(\frac{y}{\sqrt{(y-(bR_1(n)/\lambda_n)\,z)^2-4a^2(R_1^2(n)\,z^2/\lambda_n^2)}}-1\right)\frac{dz}{z}. \tag{3.40}$$

Proof. The result follows from Theorem 2.4 and Lemmas 3.1, 3.3, 3.5, and 3.7. ∎

THEOREM 3.9. *Suppose* (1.2) *and* (1.4) *are satisfied with $\alpha=0$. Suppose $a_i\to\infty$, $|b_i|\to\infty$, and* (2.8) *and* (3.5) *hold. Suppose $R_1(n)$ and $R_2(n)$ where $a_n=aR_1(n)$ and $b_n=bR_2(n)$ have C^1 extensions such that* (2.16), (3.18), *and* (3.30) *hold. Then*

$$\lim_{n\to\infty} \frac{p_n(\lambda_n y)}{(u(R_1(n)/\lambda_n))^n\, K(n)\, A(n)(a_1/a\lambda_n)^{1/2}} = \frac{R\sqrt{au(R_1(n)/\lambda_n)}}{((y-b)^2-4a^2)^{1/4}} \exp\left\{b\left(B+\frac{1}{2}\right)\int_0^1 \frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}}\right\},$$

uniformly on compact subsets of $\mathbb{C}\setminus[D, E]$. Here R, $K(n)$, $A(n)$, $u(R_1(n)/\lambda)$ are as in the previous theorem. If $R_1(x)$ has an inverse for $x \geqslant 0$, then (3.21) *may be used in place of* (3.20). *If $R_1^*(x)$, defined for $x \geqslant 0$, is an asymptotic inverse of $R_1(x)$ such that $\lim_{x\to\infty}|R_1^*(R_1(x))-x|=0$, then $K(n)$ may be replaced by $K^*(n)$ where $K^*(n)$ is given by* (3.40).

Proof. The result follows from Theorem 2.4 and Lemmas 3.2, 3.3, 3.5, and 3.7. ∎

4. Examples

As a first example we take $a_n = an^\alpha$, $n \geq 1$, $b_n = bn^\alpha$, $n \geq 0$, and $\lambda_n = n^\alpha$. In this case $a_n/a\lambda_n = 1$ and $u(R_1(n)/\lambda_n) = u(1) = (y-b)/2a + \sqrt{(y-b)/2a)^2 - 1}$. From (3.20) we see that $K(n) = H(n)\, e^{-n\alpha}$ where

$$H(n) = \exp\left(ny\alpha \int_0^1 \frac{1}{\sqrt{(y - bz^\alpha)^2 - 4a^2 z^{2\alpha}}}\, dz\right).$$

Instead of using Lemma 3.7 we can evaluate directly

$$\prod_{i=1}^{n} \frac{a_i}{a\lambda_n} = \prod_{i=1}^{n} \frac{i^\alpha}{n^\alpha} = \frac{(2\pi n)^{\alpha/2}}{e^{n\alpha}} (1 + o(1)).$$

Consequently from Theorem 3.8 we find

$$\lim_{n\to\infty} \frac{(2\pi n)^{\alpha/2} p_n(n^\alpha y)}{u(1)^n H(n)} = \frac{\sqrt{au(1)}}{((y-b)^2 - 4a^2)^{1/4}} \exp\left(\frac{b}{2}\int_0^1 \frac{dz}{\sqrt{(y-bz)^2 - 4a^2 z^{2\alpha}}}\right),$$

which was first found by Van Assche and Geronimo [17].

For a second example we consider $a_n = a\ln(n+1)$, $n \geq 1$, $b_n = b\ln(n+1)$, $n \geq 0$, and $\lambda_n = \ln(n+1)$. Again, $a_n/a\lambda_n = 1$ and $u(R_1(n)/\lambda_n) = u(1)$. From (3.21) with $R_1(0)$ replaced by $R_1(1)$ we find

$$K(n) = \exp \int_{\ln 2/\ln(n+1)}^1 (e^{\ln(n+1)z} - 1)\left(\frac{y}{\sqrt{(y-bz)^2 - 4a^2 z^2}} - 1\right)\frac{dz}{z} = H(n)/A(n),$$

where

$$H(n) = \exp\left(y \int_{\ln 2/\ln(n+1)}^1 (e^{\ln(n+1)z} - 1)\frac{1}{\sqrt{(y-bz)^2 - 4a^2 z^2}}\frac{dz}{z}\right), \tag{4.1}$$

and $A(n)$ is given by (3.38). If we use the Abel–Plana formula for $\sum_{i=1}^n \ln\ln(i+1)$ (Olver [12, pp. 290–291]), then we find that

$$R = \exp\left(-\frac{1}{2}\int_1^\infty (B_2 - B_2(x - [x]))\frac{d^2}{dx^2}\ln R_1(x)\, dx\right) = \exp\left(2\int_0^\infty \frac{\operatorname{Im} \ln\ln(2+iy)}{e^{2\pi y} - 1}\, dy\right).$$

Theorem 3.9 thus yields

$$\lim_{n\to\infty} \frac{\sqrt{\ln n}\,p_n(\lambda_n y)}{u(1)^n H(n)} = \frac{R\sqrt{a_1 u(1)}}{((y-b)^2-4a^2)^{1/4}} \exp\left(\frac{b}{2}\int_0^1 \frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}}\right). \tag{4.2}$$

As a final example we consider for $0<\alpha<1$, $a_n = a\lambda_n$, $n \geqslant 1$, $b_n = b\lambda_n$, $n \geqslant 1$ where $\lambda_n = n\exp(\ln n)^\alpha$. In this case $R_1(x) = x\exp[(\ln x)^\alpha]$, and $K(n)$ and $A(n)$ are given by (3.20) and (3.28), respectively. By comparison with the Abel–Plana formula for $\sum_{i=2}^n [\ln i + (\ln i)^\alpha]$ we see that

$$\begin{aligned} R &= \exp\left(-\frac{1}{2}\int_1^\infty (B_2 - B_2(x-[x]))\frac{d^2}{dx^2}\ln R_1(x)\,dx\right) \\ &= \exp\left(2\int_0^\infty \frac{\operatorname{Im}(\ln(2+iy)+(\ln(2+iy))^\alpha)}{e^{2\pi y}-1}\,dy\right). \end{aligned} \tag{4.3}$$

Combining gives

$$\frac{\sqrt{\lambda_n}\,p(\lambda_n y)}{u(1)^n H(n)} = \frac{R\sqrt{a_1 u(1)}}{((y-b)^2-4a^2)^{1/4}} \exp\left(\frac{b}{2}\int \frac{dz}{\sqrt{(y-bz)^2-4a^2z^2}}\right), \tag{4.4}$$

where

$$\begin{aligned} H(n) = \exp \int_0^n & \frac{y}{[(y-bx\exp(\ln x)^\alpha/\lambda_n)^2 - 4a^2(x\exp(\ln x)^\alpha/\lambda_n)^2]^{1/2}} \\ & \times (1+\alpha(\ln x)^{\alpha-1})\,dx. \end{aligned} \tag{4.5}$$

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